To sum up, we have demonstrated the cases when we can pass from the functional equations for $C_1 = (u_1, r_1)$ to a system of ordinary (not differential) equations with only a few unknowns. It can be said in general that the passage can be made if, in the expansion $C_1 = \Sigma A_k \cos(\theta_k t + \psi_k)$, the condition $|\varepsilon A_k| \geqslant \theta_k$ is satisfied by only a few harmonics. The stationary-phase method also simplifies the functional problem. Given these possibilities, our scheme is preferable to the methods described in /1/, in which the results are stated as first-order non-linear equations for the amplitudes and phases.

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PMM U.S.S.R., Vol.53, No.4, pp.439-447, 1989
Printed in Great Britain
0021-8928/89 $10.00+0.00

THE CONDITION FOR SIGN-DEFINITENESS OF INTEGRAL QUADRATIC FORMS AND THE STABILITY OF DISTRIBUTED-PARAMETER SYSTEMS*

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The stability of distributed-parameter systems described by linear partial differential equations is investigated by reducing the original equations by a change of variables to a system of first-order equations in time and in spatial coordinates. The Lyapunov functions are constructed in the form of single integral forms. New necessary and sufficient conditions for the sign-definiteness of these forms are obtained. These conditions, unlike the Sylvester criterion, do not require the calculation of determinants. The check for sign-definiteness is made using recurrence relationships and is a generalization of the results obtained in /1/.

The proposed criteria are applied to derive sufficient conditions for the stability of distributed-parameter linear systems. The construction of functionals for the one-dimensional second-order linear hyperbolic equation is considered in more detail. As an example, we examine the stability of the torsional oscillations of an aircraft wing.

1. Consider a system of first-order linear partial differential equations of the form

$$\frac{\partial \varphi}{\partial t} = \sum_{k=1}^{s} \left(A_k(\mathbf{x}) \frac{\partial \varphi}{\partial x_k} + B_k(\mathbf{x}) \frac{\partial \psi}{\partial x_k} \right) + A_0(\mathbf{x}) \varphi + B_0(\mathbf{x}) \psi \tag{1.1}$$

$$\sum_{k=1}^{s} \left(C_k(\mathbf{x}) \frac{\partial \varphi}{\partial x_k} + D_k(\mathbf{x}) \frac{\partial \psi}{\partial x_k} \right) + C_0(\mathbf{x}) \varphi + D_0(\mathbf{x}) \psi = 0 \tag{1.2}$$

where $t \in I = (0, \infty)$, $\mathbf{x} = (x_1, x_2, \ldots, x_s)^T \in X \subset E^s$ is a vector of spatial coordinates, $\varphi = \varphi(\mathbf{x}, t)$ is the n-dimensional vector of phase functions, $\psi = \psi(\mathbf{x}, t)$ is the m-dimensional vector of phase functions whose derivative with respect to time does not occur in the system (1.1), (1.2), $A_k(\mathbf{x}), B_k(\mathbf{x}), C_k(\mathbf{x})$, and $D_k(\mathbf{x})$ $(k = 0, 1, \ldots, s)$ are matrices whose elements

*Prikl. Matem. Mekhan., 53, 4, 567-575, 1989 are bounded measurable functions.

Note that any linear partial differential equation of any order or any system of such equations can be reduced to the form (1.1), (1.2) by introducing supplementary variables.

For example, the scalar hyperbolic equation

$$\frac{\partial^2 y}{\partial t^2} = a_1(x)\frac{\partial^2 y}{\partial x^2} + a_2(x)\frac{\partial y}{\partial x} + a_3(x)\frac{\partial y}{\partial t} + a_4(x)\,y \tag{1.3}$$
$$x \in (0, l), \quad a_1(x) \geqslant \text{const} > 0$$

can be reduced to the form (1.1), (1.2) by taking the function $y = y(x, t)$ and its first derivatives as the new variables:

$$y = \varphi_1, \quad \partial y/\partial t = \varphi_2, \quad \partial y/\partial x = \varphi_3 \tag{1.4}$$

We will rewrite the original Eq.(1.3) in these variables, augmenting it with integrability conditions /2/ and relationships that are obtained from (1.4) when y is eliminated. We obtain the system

$$\frac{\partial\varphi_1}{\partial t} = \varphi_2, \quad \frac{\partial\varphi_2}{\partial t} = a_1\frac{\partial\varphi_3}{\partial x} + a_2\varphi_3 + a_3\varphi_2 + a_4\varphi_1, \quad \frac{\partial\varphi_3}{\partial t} = \frac{\partial\varphi_2}{\partial x}, \quad \frac{\partial\varphi_1}{\partial x} = \varphi_3 \tag{1.5}$$

which is equivalent to Eq.(1.3). Using the notation $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T$,

$$A_1 = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & a_1 \\ 0 & 1 & 0 \end{Vmatrix}, \quad A_0 = \begin{Vmatrix} 0 & 1 & 0 \\ a_4 & a_3 & a_2 \\ 0 & 0 & 0 \end{Vmatrix}, \quad C_1 = (1, 0, 0), \quad C_0 = (0, 0, 1) \tag{1.6}$$

we rewrite this system in the form (1.1), (1.2), where $k = 1,\ x_1 = x,\ B_k = D_k = 0\ (k = 0, 1, \ldots, s)$.

In order to reduce a high-order linear partial differential equation to the form (1.1), (1.2), we should use the corresponding low-order derivatives as the supplementary variables, expressing the original equation and the integrability conditions in terms of these derivatives. The variables $\boldsymbol{\psi}$ appear if the derivatives with respect to t and $\mathbf{x}$ in the original equation are of different orders, and the number of variables is not necessarily equal to the number of equations in the system. These topics were considered in more detail in /3, 4/.

The components of the initial values of the vector function $\boldsymbol{\varphi}(\mathbf{x}, t)$ belong to the space $L_2(X)$, and the boundary conditions are defined on some part S_0 of the boundary S of the region X in the form

$$\alpha\boldsymbol{\varphi}(\mathbf{x}, t) = 0, \quad \beta\boldsymbol{\psi}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in S_0 \subset S \tag{1.7}$$

where α, β are matrices whose elements are bounded measurable functions.

The solution of system (1.1), (1.2) is considered in the class of functions from the space

$$W_2^1(X \times I) = \{\varphi_i, \psi_i \mid \varphi_i \in L_2(X \times I), \quad \psi_i \in L_2(X \times I)\}$$
$$\frac{\partial\varphi_i}{\partial t} \in L_2(X \times I), \frac{\partial\varphi_i}{\partial x_k} \in L_2(X \times I), \quad \frac{\partial\psi_i}{\partial x_k} \in L_2(X \times I), \quad \{k = 1, 2, \ldots, s\}$$

Here and in what follows, $i = 1, 2, \ldots, n$, unless otherwise specified.

Consider the stability of the solutions $\boldsymbol{\varphi} \equiv \boldsymbol{\psi} \equiv 0$ of system (1.1), (1.2), (1.7) in the measure

$$\rho = \int_X \boldsymbol{\varphi}^T(\mathbf{x}, t)\,\boldsymbol{\varphi}(\mathbf{x}, t)\,d\mathbf{x} \tag{1.8}$$

The Lyapunov function is constructed as the integral quadratic form

$$V = \int_X \boldsymbol{\varphi}^T(\mathbf{x}, t)\,v(\mathbf{x})\,\boldsymbol{\varphi}(\mathbf{x}, t)\,d\mathbf{x} \tag{1.9}$$

where $v(\mathbf{x})$ is a symmetrical matrix whose elements are bounded functions differentiable almost everywhere on X.

We find the derivative of the function V by Eq.(1.1),

$$\frac{dV}{dt}=\int\limits_X\Big[\sum_{k=1}^{s}\Big(\varphi^T vA_k\frac{\partial\varphi}{\partial x_k}+\frac{\partial\varphi^T}{\partial x_k}A_k{}^T v\varphi+\varphi^T vB_k\frac{\partial\psi}{\partial x_k}+ \tag{1.10}$$
$$\frac{\partial\psi^T}{\partial x_k}B_k{}^T v\varphi\Big)+\varphi^T(vA_0+A_0{}^T v)\varphi+\varphi^T vB_0\psi+\psi^T B_0{}^T v\varphi\Big]\,d\mathbf{x}$$

Using Eq.(1.2), we supplement this expression with the equality

$$\int\limits_X\Big\{(\varphi^T\Gamma_1+\psi^T\Gamma_2)\Big[\sum_{k=1}^{s}\Big(C_k\frac{\partial\varphi}{\partial x_k}+D_k\frac{\partial\psi}{\partial x_k}\Big)+C_0\varphi+D_0\psi\Big]+$$
$$\Big[\sum_{k=1}^{s}\Big(\frac{\partial\varphi^T}{\partial x_k}C_k{}^T+\frac{\partial\psi^T}{\partial x_k}D_k{}^T\Big)+\varphi^T C_0{}^T+\psi^T D_0{}^T\Big](\Gamma_1{}^T\varphi+\Gamma_2{}^T\psi)\Big\}\,d\mathbf{x}=0$$

where $\Gamma_1=\Gamma_1(\mathbf{x})$ and $\Gamma_2=\Gamma_2(\mathbf{x})$ are matrices (as yet arbitrary) with elements from the space of functions differentiable almost everywhere on X. Integrating by parts, we obtain

$$\frac{dV}{dt}=\int\limits_X\Big\{-\varphi^T\omega\varphi+\psi^T\Big[-\sum_{k=1}^{s}\frac{\partial(\Gamma_2D_k)}{\partial x_k}+\Gamma_2D_0+D_0{}^T\Gamma_2\Big]\psi+$$
$$\varphi^T\Big[-\sum_{k=1}^{s}\frac{\partial(vB_k+\Gamma_1D_k)}{\partial x_k}+vB_0+\Gamma_1D_0+C_0{}^T\Gamma_2{}^T\Big]\psi+$$
$$\psi^T\Big[-\sum_{k=1}^{s}\frac{\partial(B_k{}^Tv+D_k{}^T\Gamma_1{}^T)}{\partial x_k}+B_0{}^Tv+D_0{}^T\Gamma_1{}^T+\Gamma_2C_0\Big]\varphi+$$
$$\sum_{k=1}^{s}\Big[\frac{\partial\varphi^T}{\partial x_k}(A_k{}^Tv+C_k{}^T\Gamma_1{}^T-vA_k-\Gamma_1C_k)\varphi+\frac{\partial\psi^T}{\partial x_k}(D_k{}^T\Gamma_2{}^T-\Gamma_2D_k)\psi+$$
$$\frac{\partial\varphi^T}{\partial x_k}(C_k{}^T\Gamma_2{}^T-vB_k-\Gamma_1D_k)\psi+\psi^T\Big(\Gamma_2C_k-B_k{}^Tv-D_k{}^T\Gamma_1{}^T\Big)\frac{\partial\varphi}{\partial x_k}\Big]\Big\}\,d\mathbf{x}+$$
$$\int\limits_S\Big[\sum_{k=1}^{s}(\varphi^T(vA_k+\Gamma_1C_k)\varphi+\psi^T\Gamma_2D_k\psi+\varphi^T(vB_k+\Gamma_1D_k)\psi+$$
$$\psi^T(B_k{}^Tv+D_k{}^T\Gamma_1{}^T)\varphi)\cos(n,x_k)\Big]\,d\mathbf{x}$$

Here

$$\omega=\sum_{k=1}^{s}\frac{\partial(vA_k+\Gamma_1C_k)}{\partial x_k}-vA_0-A_0{}^Tv-\Gamma_1C_0-C_0{}^T\Gamma_1{}^T,\quad x\in^{\cdot}X \tag{1.11}$$

where n is the outer normal to S; the notation $\mathbf{x}\in^{\cdot}X$ indicates that Eq.(1.11) holds almost everywhere on X.

Let the matrices Γ_1,Γ_2 satisfy the equations

$$vA_k+\Gamma_1C_k=A_k{}^Tv+C_k{}^T\Gamma_1{}^T,\quad \Gamma_2D_k=D_k{}^T\Gamma_2{}^T,\quad C_k{}^T\Gamma_2{}^T=vB_k+\Gamma_1D_k \tag{1.12}$$
$$\sum_{k=1}^{s}\frac{\partial(\Gamma_2D_k)}{\partial x_k}-\Gamma_2D_0-D_0{}^T\Gamma_2{}^T=0$$
$$\sum_{k=1}^{s}\frac{\partial(vB_k+\Gamma_1D_k)}{\partial x_k}-vB_0-\Gamma_1D_0-C_0\Gamma_2{}^T=0,\quad \mathbf{x}\in^{\cdot}X,\quad k=1,2,\ldots,s$$

$$\sum_{k=1}^{s}[\varphi^T(vA_k+\Gamma_1C_k)\varphi+\psi^T\Gamma_2D_k\psi+\varphi^T(vB_k+\Gamma_1D_k)\psi+ \tag{1.13}$$
$$\psi^T(B_k{}^Tv+D_k{}^T\Gamma_1{}^T)\varphi]\cos(n,x_k)=0,\mathbf{x}\in^{\cdot}S$$

Then for the derivative we obtain

$$\frac{dV}{dt}=-\int\limits_X\varphi^T(\mathbf{x},t)\,\omega(\mathbf{x})\,\varphi(\mathbf{x},t)\,d\mathbf{x} \tag{1.14}$$

i.e., a quadratic form similar to that for V (1.9).

By the method of Lyapunov functions /5/, the solution $\varphi = \psi \equiv 0$ of system (1.1), (1.2), (1.7) is asymptotically stable in the measure ρ (1.8) if the functional (1.9) is continuous and positive definite in the measure ρ, while its derivative dV/dt (1.14) is negative definite in this measure. In stability analysis, the condition of negative definiteness of the derivative dV/dt (1.14) is replaced with the condition of non-positive definiteness.

The continuity of the functional V (1.9) in the measure ρ (1.8) follows directly from the boundedness of the matrix $v(\mathbf{x})$. Thus, stability analysis reduces to checking integral quadratic forms (1.9) for sign-definiteness.

These results also suggest a solution for the problem of constructing the functional V (1.9) given a symmetrical matrix $\omega(\mathbf{x})$. This involves solving Eqs.(1.11), (1.12) for the matrices $v(\mathbf{x}), \Gamma_1(\mathbf{x}), \Gamma_2(\mathbf{x})$ with the boundary conditions that follow from (1.13), (1.7). However, unlike the problem of constructing quadratic forms for ordinary differential equations, not all the elements of the matrix $\omega(\mathbf{x})$ may be arbitrary in this case. This problem is considered in more detail in Sect.4 for a one-dimensional second-order hyperbolic equation.

2. Let us consider the conditions of sign-definiteness. First we will derive the necessary and sufficient conditions of sign-definiteness of the integral quadratic form

$$F = \int_X \varphi^T(\mathbf{x}) f^T(\mathbf{x}) f(\mathbf{x}) \varphi(\mathbf{x})\, d\mathbf{x} \tag{2.1}$$

in the measure ρ (1.8). Here $\varphi(\mathbf{x})$ is the n-dimensional vector function with arbitrary components $\varphi_i(\mathbf{x}) \in L_2(X)$, and $f(\mathbf{x}) = \| f_{ij}(\mathbf{x}) \|$ is a square triangular matrix whose elements are bounded measurable functions and the elements under the main diagonal are zero almost everywhere on X, i.e., $f_{ij}(\mathbf{x}) = 0,\ \mathbf{x} \in^{\cdot} X\ (j < i,\ j = 1, 2, \ldots, n-1)$.

Theorem 1. For positive definiteness of the integral quadratic form F (2.1) in the measure ρ (1.8) it is necessary and sufficient that

$$|f_{ii}(\mathbf{x})| > 0, \quad \mathbf{x} \in^{\cdot} X \tag{2.2}$$

i.e., that for any set $\tau \subset X$ of finite measure there exists a positive number ε such that

$$\int_\tau |f_{ii}(\mathbf{x})|^2\, d\mathbf{x} \geqslant \varepsilon > 0 \tag{2.3}$$

Proof. Necessity. Assume that the form F (2.1) is positive definite in ρ (1.8), i.e., if $\rho \geqslant \varepsilon > 0$, then there exists a number $\delta = \delta(\varepsilon) > 0$ such that $F \geqslant \delta(\varepsilon)$. Now, assume that for all values of the index i less than s, where $s \in \{1, 2, \ldots, n\}$ is a fixed index, we have $|f_{ii}(\mathbf{x})| > 0,\ \mathbf{x} \in^{\cdot} X$, and for $i = s$ there exists a set $\tau \subset X$ of finite measure where $f_{ss}(\mathbf{x}) = 0,\ \mathbf{x} \in^{\cdot} \tau$. The behaviour of $f_{ss}(\mathbf{x})$ and the set $X \setminus \tau$ and the behaviour of other elements $f_{ij}(\mathbf{x}$ on X is irrelevant.

Since the choice of the functions $\varphi(\mathbf{x})$ is arbitrary, we choose them so that

$$\varphi_i(\mathbf{x}) = 0, \quad \mathbf{x} \in^{\cdot} X \quad (i = s+1, s+2, \ldots, n); \quad \varphi_i(\mathbf{x}) = 0, \quad \mathbf{x} \in^{\cdot} X \setminus \tau$$
$$(i = 1, 2, \ldots, s)$$
$$\sum_{i=j}^{s} f_{ij}(\mathbf{x}) \varphi_j(\mathbf{x}) = 0, \quad \mathbf{x} \in^{\cdot} \tau \quad (i = 1, 2, \ldots, s-1)$$
$$|\varphi_s(\mathbf{x})| \geqslant \varepsilon_1 > 0, \quad \mathbf{x} \in^{\cdot} \tau_1 \subset \tau \subset X; \quad |\varphi_s(\mathbf{x})| \geqslant 0, \quad \mathbf{x} \in^{\cdot} \tau \setminus \tau_1$$

Then

$$\rho = \int_\tau \sum_{i=1}^{s} \varphi_i^2 d\mathbf{x} \geqslant \int_{\tau_1} \varphi_s^2 d\mathbf{x} \geqslant \varepsilon_1 |\tau_1| = \varepsilon > 0$$

where $|\tau_1|$ is the measure of the set τ_1. By the conditions on f_{ss} and φ_i, we obtain

$$F = \int_\tau (f_{ss}\varphi_s)^2\, d\mathbf{x} = 0$$

which contradicts the positive definiteness of the form F (2.1) in the measure ρ (1.8), i.e., for $\rho \geqslant \varepsilon > 0$ there is no number $\delta(\varepsilon) > 0$ such that $F \geqslant \delta(\varepsilon)$. Therefore, the condition $|f_{ss}(\mathbf{x})| > 0,\ \mathbf{x} \in^{\cdot} X$ is necessary. Repeating the argument for all $s = 1, 2, \ldots, n$, we prove the necessity of Conditions (2.2).

Sufficiency. Assume that Conditions (2.2) hold. We will show that the form F (2.1) is positive definite in the measure ρ (1.8), i.e, if $\rho \geqslant \varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that $F \geqslant \delta(\varepsilon)$.

Thus, let Condition (2.2) hold and let $\rho \geqslant \varepsilon > 0$. But if $\rho \geqslant \varepsilon > 0$, then for at least one value of the index i, say $i = s$, we have $|\varphi_s(\mathbf{x})| \geqslant \gamma_s(\varepsilon) > 0$ at least on some set of finite measure $\tau_s \subset X$. Indeed, if no such set exists, i.e, if $|\tau_s| \to 0$, where $|\tau_s|$ is the measure of the set τ_s, then $\rho \to 0$ by $\varphi_s \in L_2(X)$ and the absolute continuity of the Lebesgue integral /6/, which contradicts the inequality $\rho \geqslant \varepsilon > 0$.

Let us now show that there exists a number $\delta = \delta(\varepsilon) > 0$ such that $F \geqslant \delta(\varepsilon)$. First assume that $|\varphi_n(\mathbf{x})| \geqslant \gamma_n > 0$ on $\tau_n \subset X$, i.e., $s = n$. Then

$$F \geqslant \int_{\tau_n} (f_{nn}\varphi_n)^2 d\mathbf{x} \geqslant \gamma_n^2 \int_{\tau_n} f_{nn}^2 d\mathbf{x} \geqslant \gamma_n^2 \varepsilon_1 = \delta_n > 0$$

If $\varphi_n(\mathbf{x}) = 0$, $\mathbf{x} \in^{\cdot} X$, but $|\varphi_{n-1}(\mathbf{x})| \geqslant \gamma_{n-1} > 0$ on $\tau_{n-1} \subset X$, then we again obtain

$$F \geqslant \int_{\tau_{n-1}} (f_{n-1,\,n-1}\varphi_{n-1})^2 d\mathbf{x} \geqslant \delta_{n-1} > 0$$

Continuing this reasoning step by step for $s = n-2, n-3, \ldots, 1$, we show that if $\rho \geqslant \varepsilon > 0$, then $F \geqslant \delta(\varepsilon) > 0$, where $\delta(\varepsilon) = \min(\delta_1, \delta_2, \ldots, \delta_n)$, i.e., F is a positive definite form. The theorem is proved.

Now consider the integral quadratic form

$$V = \int_X \varphi^T(\mathbf{x}) v(\mathbf{x}) \varphi(\mathbf{x})\, d\mathbf{x} \tag{2.4}$$

where $\varphi(\mathbf{x})$ is a n-dimensional vector function with arbitrary components $\varphi_i \in L_2(X)$, and $v(\mathbf{x}) = \| v_{ij}(\mathbf{x}) \|$ is a symmetrical matrix whose elements are bounded measurable functions. Let

$$v_x = \varphi^T(\mathbf{x}) v(\mathbf{x}) \varphi(\mathbf{x}) \tag{2.5}$$

Theorem 2. For positive definiteness of the integral quadratic form V (2.4) in the measure ρ (1.8) it is necessary and sufficient that

$$v_{ii}(\mathbf{x}) - \sum_{j=1}^{i-1} b_{ji}^2(\mathbf{x}) > 0, \quad \mathbf{x} \in^{\cdot} X \tag{2.6}$$

where the functions $b_{ji}(\mathbf{x})$ $(j \leqslant i,\ j = 1, 2, \ldots, n)$ are computed from the recurrence formulas

$$b_{ii}(\mathbf{x}) = \pm \left[v_{ii}(\mathbf{x}) - \sum_{k=1}^{i-1} b_{ki}^2(\mathbf{x}) \right]^{1/2}, \quad \mathbf{x} \in^{\cdot} X \tag{2.7}$$

$$b_{ij}(\mathbf{x}) = \frac{1}{b_{ii}(\mathbf{x})} \left[v_{ij}(\mathbf{x}) - \sum_{k=1}^{i-1} b_{ki}(\mathbf{x}) b_{kj}(\mathbf{x}) \right], \mathbf{x} \in^{\cdot} X \quad (j = i+1, i+2, \ldots, n) \tag{2.8}$$

$$b_{ij}(\mathbf{x}) = 0, \quad \mathbf{x} \in^{\cdot} X \quad (i > j,\ j = 1, 2, \ldots, n-1)$$

Proof. *Necessity*. Assume that the form (2.4) is positive definite in the measure ρ (1.8). Then for a given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $V > \delta(\varepsilon)$ if $\rho > \varepsilon$. First we will show that in this case the quadratic form (2.5) is positive definite for $\mathbf{x} \in^{\cdot} X$, i.e., $v_x > 0$ for $\varphi^T\varphi > 0$ almost everywhere on X.

Assume that this is not so, i.e., $v_x \leqslant 0$ at least on some set $\tau \subset X$ of finite measure $|\tau|$. Since $\varphi(\mathbf{x})$, and therefore $\varphi^T\varphi$, is arbitrary, let $\varphi^T\varphi > 0$ on $\tau \subset X$ and $\varphi^T\varphi = 0$ on $X \setminus \tau$ when $\rho \geqslant \varepsilon > 0$. Then $V = \int_\tau v_x d\mathbf{x} \leqslant 0$, which contradicts the positive definiteness of V (2.4) in ρ (1.8). Hence it follows that the form v_x (2.5) is also positive definiteness almost everywhere on X.

The positive definite form v_x (2.5) for a fixed $\mathbf{x} \in^{\cdot} X$ is representable as

$$v_x = \varphi^T(\mathbf{x}) B^T(\mathbf{x}) B(\mathbf{x}) \varphi(x) \tag{2.9}$$

where $B(\mathbf{x}) = \| b_{ij}(\mathbf{x}) \|$ is a triangular matrix with zeros under the main diagonal. We have the inequalities

$$|b_{ii}(\mathbf{x})| > 0, \quad \mathbf{x} \in X \tag{2.10}$$

Indeed, if $b_{ii}(\mathbf{x}) = 0$ on a set $\tau \subset X$ of finite measure, then by Theorem 1 the form V (2.4) with the integrand function represented by (2.9) cannot be positive definite in the measure ρ (1.8). We have obtained a contradiction. Thus, inequalities (2.10) hold.

Now, comparing (2.5) and (2.9), we obtain $v(\mathbf{x}) = B^T(\mathbf{x}) B(\mathbf{x})$, or in scalar notation

$$v_{ii}(\mathbf{x}) = \sum_{k=1}^{i} b_{ki}^2(\mathbf{x}), \quad v_{ij}(\mathbf{x}) = \sum_{k=1}^{i} b_{ki}(\mathbf{x}) b_{kj}(\mathbf{x}) \quad (j = i+1, i+2, \ldots, n) \tag{2.11}$$

Solving system (2.11) for various values of the indices i and j, starting with $i = 1$, $j = 1$, we obtain (2.7) and (2.8). Substituting (2.7) into (2.10), we verify that inequalities (2.6) hold. This completes the proof of the necessity of Conditions (2.6).

Sufficiency. Assume that the inequalities (2.6) with the functions $b_{ij}(\mathbf{x})$ calculated from (2.7), (2.8) are satisfied. Then the given quadratic form V (2.4) is representable in the form

$$V = \int_X \varphi^T(\mathbf{x}) B^T(\mathbf{x}) B(\mathbf{x}) \varphi(\mathbf{x}) \, d\mathbf{x}$$

and by (2.6), (2.7) it satisfies Conditions (2.10). Therefore, by Theorem 2, the form V (2.4) is positive definite in the measure ρ (2.2). The theorem is proved.

Note that the proof of the necessity of the conditions of Theorem 1 and 2 essentially uses the fact that the functions φ_i are arbitrary. If not all these functions are independent, then (2.3) and (2.6) are only the sufficient conditions for positive definiteness of the integral forms (2.1) and (2.4), respectively, because the proof of sufficiency does not require the functions φ_i to be arbitrary.

For example, the functional $V = \int (\varphi_1^2 - a\varphi_2^2) \, dx$ (here and in what follows, integration over x is from 0 to 1), where $a = \text{const} > 0$, $\varphi_1 = \partial\varphi_2/\partial x$, $\varphi_2(0, t) = 0$, has the bound (we use the inequality /5/ $\int \varphi_2^2 dx < {}^1/_2 \int \varphi_1^2 dx$)

$$V = \int (\beta\varphi_1^2 + (1-\beta)\varphi_1^2 - a\varphi_2^2) \, dx \geqslant \int (\beta\varphi_1^2 + (2(1-\beta) - a)\varphi_2^2) \, dx$$

where β is a number such that $0 < \beta < 1$. We choose β so that $\beta = 2(1-\beta) - a$. Hence we obtain $\beta = {}^1/_3(2-a)$. The condition $0 < \beta < 1$ is satisfied if $a < 2$. Here

$$V \geqslant {}^1/_3(2-a) \int (\varphi_1^2 + \varphi_2^2) \, dx$$

Thus, the functional V is positive definite in the measure $\rho = \int (\varphi_1^2 + \varphi_2^2) dx$, if $a < 2$, although the form $\varphi_1^2 - a\varphi_2^2$ is not positive definite in φ_1 and φ_2.

3. Applying these conditions of sign-definiteness of integral quadratic forms, we will derive the sufficient conditions of asymptotic stability of the solution $\varphi = \psi \equiv 0$ of system (1.1), (1.2), (1.7) in the measure ρ (1.8).

Theorem 3. Assume that the matrices $v(\mathbf{x}), \Gamma_1(\mathbf{x}), \Gamma_2(\mathbf{x})$ satisfying the Conditions (1.12), (1.13) exist and that the inequalities (2.6) and

$$\omega_{ii}(\mathbf{x}) - \sum_{j=1}^{i-1} d_{ji}^2(\mathbf{x}) > 0, \mathbf{x} \in X \tag{3.1}$$

hold, where $\omega_{ii}(\mathbf{x})$ are the elements of the matrix $\omega(\mathbf{x})$ (1.11), and the functions $d_{ji}(\mathbf{x})$ $(j \leqslant i, \ j = 1, 2, \ldots, n)$ are computed from recurrences similar to (2.7), (2.8) with v_{ij} replaced by ω_{ij}. Then the solution $\varphi = \psi \equiv 0$ of system (1.1), (1.2), (1.7) is asymptotically stable in the measure ρ (1.8).

The proof of the theorem follows from the fact that by Theorem 2 the form V (1.9) is positive definite and the derivative dV/dt (1.14) by the equations of the process (1.1), (1.2), (1.7) is negative definite in the measure ρ (1.8).

Similar conditions of asymptotic stability of the solution $\varphi \equiv \psi \equiv 0$ in the measure ρ (1.8) can be formulated using the integral quadratic form (2.1). In this case, by Theorem 3, inequalities (2.6) are replaced by the simpler inequalities (2.2), and the matrix $v(\mathbf{x})$ in (1.11)-(1.13) is replaced by the matrix $f^T(\mathbf{x})f(\mathbf{x})$, where $f(\mathbf{x})$ is the matrix from (2.1).

4. Consider the construction of the functional V (1.9) for Eq.(1.3) with linear homogeneous boundary conditions, e.g, of the form

$$b_1 y + b_2 \partial y/\partial x + b_3 \partial y/\partial t = 0;\quad x = 0;\quad l,\ t \geqslant 0 \qquad (4.1)$$
$$b_1^2 + b_2^2 + b_3^2 \neq 0$$

where b_1, b_2, and b_3 are constants. Introducing new variables, we replace Eq.(1.3), (4.1) with the equivalent system (1.5) and

$$b_1\varphi_1 + b_2\varphi_3 + b_3\varphi_2 = 0;\quad x = 0;\quad l,\ t \geqslant 0 \qquad (4.2)$$

In this case, the matrices B_K, D_K, Γ_2 in Eqs.(1.12) are zero. Therefore, only the first of these equations remains, which together with (1.11), where $k = 1$ and $x_1 = x$, can be written in scalar form. This gives the following system of finite and ordinary differential equations:

$$\gamma_{21} = v_{13},\ \gamma_{31} = a_1 v_{12},\ a_1 v_{22} = v_{33} \qquad (4.3)$$
$$d\gamma_{11}/dx - 2a_4 v_{12} = \omega_{11},\quad d(a_1 v_{23})/dx + 2a_1 v_{12} - 2a_2 v_{23} = \omega_{33}$$

$$d(a_1 v_{12})/dx - a_2 v_{12} - a_4 v_{23} + \gamma_{11} = \omega_{13},\quad d(a_1 v_{22})/dx - a_2 v_{22} - a_3 v_{23} = \omega_{23}, \qquad (4.4)$$
$$dv_{13}/dx - v_{11} - a_3 v_{12} - a_4 v_{22} = \omega_{12},\quad dv_{23}/dx - 2v_{12} - 2a_3 v_{22} = \omega_{22}$$

where $v_{ij} = v_{ij}(x),\ \gamma_{i1} = \gamma_{i1}(x),\ \omega_{ij} = \omega_{ij}(x)$ are the elements of the matrices $v(x)$, $\Gamma_1(x)$, and $\omega(x)$, respectively.

Condition (1.13) is rewritten as

$$\varphi^T(l, t)[v(l)A_1(l) + \Gamma_1(l)C_1(l)]\varphi(l, t) - \varphi^T(0, t)[v(0)A_1(0) + \qquad (4.5)$$
$$\Gamma_1(0)C_1(0)]\varphi(0, t) = 0$$

In order to construct the functional V, we have to solve system (4.4) for γ_{11}, v_{ij}, with the boundary conditions that follows from (4.5) subject to Conditions (4.2) and relationships (4.3), using given values of the functions ω_{ij}. However, as we see from Eqs.(4.4), the functions ω_{22} and ω_{33} cannot always be specified independently of one another, because the first and the fourth equations in (4.4) are consistent only if ω_{22} and ω_{33} satisfy the relationship

$$\omega_{22} = (\omega_{33} - ((da_1/dx) - 2a_2)v_{23} - 4a_1 v_{12} - 2a_1 a_3 v_{22})/a_1 \qquad (4.6)$$

In many cases, ω_{22} is determined from Eq.(4.6) apart from an arbitrary constant, so that its value can be varied.

Note that the functions v_{11} and v_{13} only occur in the last equation but one in (4.4). In order to determine these functions separately, one of them is chosen arbitrarily but satisfying the boundary conditions, if given.

Thus, the problem of constructing the functional V is solved in the following order:

1) specify the values $\omega_{11}, \omega_{12}, \omega_{13}, \omega_{23}, \omega_{33}$;

2) solve the first four equations in (4.4) with the boundary conditions that follow from (4.5), (4.2), (4.3) to determine $\gamma_{11}, v_{12}, v_{23}, v_{22}$;

3) find v_{33}, as it follows from (4.3), using the formula $v_{33} = a_1 v_{22}$;

4) from the fifth equation in (4.4) find the functions v_{11} or v_{13}, choosing one of them arbitrarily;

5) from Eq.(4.6) compute ω_{22}.

The functional V (1.9) and its derivative dV/dt (1.14), by virtue of (1.5) and (4.2), are completely determined.

Example. Consider the stability of torsional oscillations of an aircraft wing, described by the equations

$$Il^2\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(R\frac{\partial y}{\partial x}\right) - hl^2\frac{\partial y}{\partial t} - Ml^2 y,\quad x \in (0, 1) \qquad (4.7)$$
$$y(0, t) = \partial y/\partial x\,|_{x=1} = 0$$

where l is the wing half-span, x is the coordinate normalized by l, $y = y(x, t)$ is the torsion angle of the cross-section with the coordinate x; $I = I(x)$, $R = R(x)$, $h = h(x)$ are the linear moment of inertia, the torsional stiffness, and the damping ratio in this cross-section, respectively; the term $M(x)l^2y$ is the linear moment of the aerodynamic forces.

Eqs.(4.7) are written in the form of the system (1.5), where

$$a_1 = R/(Il^2),\ a_2 = (dR/dx)/(Il^2),\ a_3 = -h/I,\ a_4 = -M/I \tag{4.8}$$

and

$$\varphi_1(0, t) = \varphi_3(1, t) = 0 \tag{4.9}$$

From (4.9) we obtain

$$\partial\varphi_1(0, t)/\partial t = \varphi_2(0, t) = 0 \tag{4.10}$$

Using (4.9), (4.10) and (4.3), we obtain from (4.5)

$$\gamma_{11}(1) = v_{23}(0) = v_{23}(1) = v_{13}(1) = 0 \tag{4.11}$$

Let us construct the functional V. Take $\omega_{12} = \omega_{13} = \omega_{23} = 0,\ \omega_{11} = -2a_4v_{12},\ \omega_{33} = 2a_1v_{12}$. Solving the system of the first four equations from (4.4) with the boundary Conditions (4.11) and using the values of $a_1, \ldots, a_4$ (4.8), we obtain

$$\gamma_{11} = 0,\ v_{12} = c_1Il^2,\ v_{23} = 0,\ v_{22} = c_2Il^2,\ v_{33} = a_1v_{22} = c_2R$$

where c_1, c_2 are arbitrary constants. Therefore, $\omega_{11} = 2c_1Ml^2,\ \omega_{23} = 2c_1R$. Setting $v_{13} = 0$, we obtain from the fifth equation in (4.4) $v_{11} = -a_3v_{12} - a_4, v_{22} = (c_1h + c_2M)l^2$.

From Eq.(4.6) we obtain $\omega_{22} = 2(c_2h - c_1I)l^2$.

Let $c_1 = 1$. Then the functional V (1.9) and its derivative dV/dt (1.14) by system (1.5), (4.8), (4.9) are written in the form (integration over x is from 0 to 1 throughout)

$$V = \int [(h + c_2M)l^2\varphi_1^2 + 2Il^2\varphi_1\varphi_2 + c_2Il^2\varphi_2^2 + c_2R\varphi_3^2]\,dx \tag{4.12}$$

$$dV/dt = -2\int [Ml^2\varphi_1^2 + (c_2h - I)l^2\varphi_2^2 + R\varphi_3^2]\,dx \tag{4.13}$$

Using the inequality /1/ $\int \varphi_1^2dx \leqslant {}^1/_2 \int \varphi_3^2lx$, we obtain the bounds

$$V \geqslant \int \{[(h + c_2M)l^2 + 2c_2(1-\theta_1)\min R]\varphi_1^2 + c_2Il^2\varphi_2^2 + 2Il^2\varphi_1\varphi_2 + c_2\theta_1 \min R\varphi_3^2\}\,dx \tag{4.14}$$

$$dV/dt \leqslant -2\int \{[Ml^2 + 2(1-\theta_2)\min R]\varphi_1^2 + (c_2h - I)l^2\varphi_2^2 + \theta_2 \min R\varphi_3^2\}\,dx \tag{4.15}$$

where θ_1 and θ_2 are arbitrary numbers such that $0 < \theta_1 < 1,\ 0 < \theta_2 < 1$, and the min operation is over all $x \in [0, 1]$.

From inequality (4.15), (3.1) we obtain that the form dV/dt (4.13) is negative definite in the measure

$$\rho = \int (\varphi_1^2 + \varphi_2^2 + \varphi_3^2)\,dx \tag{4.16}$$

if

$$2\min R + \min Ml^2 > 0,\quad c_2 > \max I/\min h \tag{4.17}$$

The conditions of positive definiteness of the functional V (4.12) in this measures are written by the criterion (2.6) as

$$\begin{gathered} c_2 > 0,\ c_2(2\min R + \min Ml^2) + \min hl^2 > 0 \\ c_2[c_2(2\min R + \min Ml^2) + \min hl^2] > \max Il^2 \end{gathered} \tag{4.18}$$

The first two inequalities in (4.18) follow directly from inequalities (4.17), and the last inequality is obtained from (4.17) by representing it in the form

$$c_2^2(2\min R + \min Ml^2) \geqslant l^2(\max I - c_2\min h)$$

Here by (4.17) the left-hand side is positive and the right-hand side is negative.

Thus, the solution $\varphi \equiv 0$ of system (1.5), (4.8), (4.9) is asymptotically stable in the measure ρ (4.16) if the first inequality in (4.17) holds. The second inequality in (4.17) is not a stability condition. It is the condition that the constant c_2 should satisfy in order to ensure that V (4.12) and dV/dt (4.13) are sign-definite in the measure ρ (4.16).

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Translated by Z.L.

PMM U.S.S.R., Vol.53, No.4, pp.447-452, 1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

THE USE OF HIGH-ORDER FORMS IN STABILITY ANALYSIS*

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A method is proposed for determining whether forms of arbitrary high order are positive or negative definite in a region of R^n coinciding with one of the coordinate angles. Using such functions, one can then establish various modifications of well-known results of stability theory. A theorem of Grujic /1/ concerning the exponential stability of large-scale systems, generalized to the case of m-th order estimates, yields new zones of absolute stability for the equations of translational motion of an aircraft. Various results are established pertaining to the monotone stability of systems in which the right-hand side is a polynomial of a special kind.

In many problems of stability theory it suffices to construct a Lyapunov function which is positive or negative definite not in the whole space but only in a subspace, namely, a cone. This is a logical approach, for example, in relation to biological communities, since the trajectories of a system describing the dynamics of such interactions never leave the first orthant. Conditions for quadratic forms to be positive (negative) definite in a specific cone - one of the coordinate angles - were studied in /2/. A criterion for a quadratic form to be positive (negative) definite in a certain region of R^n, similar in a sense to the conditions obtained in /2/, was established in /3/ and /4/. Even before that, a criterion was proposed /5/ for a form of order 3 to be positive (negative) definite in one of the coordinate angles. Also worthy of mention is a method described in /6/ to determine whether forms of arbitrary even order are definite in the whole space.

Relying on the concept of a cone coinciding with a coordinate angle, as well as the results and /5/ and /6/, a method can be devised to investigate whether a form of arbitrary high (including odd) order is definite in an orthant of R^n.

1. Definiteness of a form of arbitrary order m in a cone. A cone in R^n coinciding with a coordinate angle will be denoted as follows /7/: $K\{\alpha_{10}, \ldots, \alpha_{n0}\}$, where α_{i0} are elements of a basis $\{\alpha_{i0}\}$ taking values +1 and -1, and

$$\alpha_{i0} = \operatorname{sign} x_i, \quad x_i \neq 0; \quad \alpha_{i0} x_i > 0$$

Throughout, $i = 1, 2, \ldots, n$.

In a cone K of the region $H = \{x: 0 \leqslant \| x \| = |x_1| + \ldots + |x_n| < \infty\}$ we consider an m-th order form

$$W(x) = \sum_{i_1=1}^{n} \cdots \sum_{i_m=i_{m-1}}^{n} A_{i_1 \ldots i_m} x_{i_1} \cdots x_{i_m}, \quad A_{i_1 \ldots i_m} = \text{const}$$

*Prikl. Matem. Mekhan., 53, 4, 576-581, 1989